# Lyapunov-type inequalities for $\psi-$Laplacian equations

Jun Zheng[1], Xu Guo[2]

[1]*School of Mathematics, Southwest Jiaotong University*
*Chengdu, Sichuan, P. R. of China 611756*
[2]*School of Information Science & Technology, Southwest Jiaotong University*
*Chengdu 611756, Sichuan, China*



**Abstract**

In this work, we present several Lyapunov-type inequalities for a class of $\psi-$Laplacian equations of the form

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0,$$

with Dirichlet boundary conditions, where $\psi$ and $f$ satisfies certain structural conditions with general nonlinearities. We do not require any sub-multiplicative property of $\psi$, and any convexity of $\frac{1}{\psi(t)}$ or $\psi(t)t$ in the establishment of Lyapunov-type inequalities. The obtained inequalities can be seen as extensions and complements of the existing results in the literature.

*Key words:* Lyapunov inequality, $\psi-$Laplacian, nonlinear equation.

## 1 Introduction

Consider the Hill's equation

$$u''(x) + r(x)u(x) = 0, \quad x \in (a, b), \tag{1a}$$
$$u(a) = u(b) = 0, \tag{1b}$$

where $r$ is a continuous and nonnegative function defined in $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. If there exists a nontrivial solution $u$ of (1), then the inequality

$$\int_a^b r(x)\mathrm{d}x \geq \frac{4}{b-a}, \tag{2}$$

holds. This result is due originally to Lyapunov [11], and is known as "Lyapunov inequality". The Lyapunov inequality and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations, and also in time scales. In the last few years independent works appeared generalizing Lyapunovs inequality for the $p$-Laplacian, by using Hölder, Jensen or Cauchy-Schwarz inequalities. A thorough literature review of Lyapunov-type inequalities and their applications can be found in the survey articles by Brown and Hinton [3], Cheng [6] and Tiryaki [18]. Some other related topics can be found in the resent articles [1,2,4,5,7–9,13–15,17,19] and the references therein.

We present some results related directly to our problem. In 2005, De Nápoli and Pinasco considered Lyapunov-type inequalities for certain nonlinear differential equations ($\psi$-Laplacian equations) generalizing the $p$-Laplacian. The main result in [12] is:

⋆ Corresponding author: J. Zheng
*Email addresses:* zhengjun@swjtu.edu.cn (Jun Zheng[1]), guoxu@my.swjtu.edu.cn (Xu Guo[2]).

**Theorem A** Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function such that $\psi(t) = t\psi(t)$ is a convex function. Moreover, suppose that there exists a constant $k > 0$ such that $\psi(2t) \leq k\psi(t)$ for any $t \geq 0$. If $r(x)$ be a positive integrable function, and the following problem

$$(\psi(u'(x)))' + r(x)\psi(u(x)) = 0 \ \text{ in } (a, b), \tag{3a}$$
$$u(a) = u(b) = 0. \tag{3b}$$

admits a nontrivial solution, then

$$2\left(\frac{k}{2}\right)^{[1-\log_2(b-a)]} \leq \int_a^b r(x)\mathrm{d}x,$$

where $[v]$ is the largest integer less than or equal to $v$.

In 2011, Sáchez and Vergara extended (3) to equations with a general nonlinear form, considering (see [16])

$$(\psi(u'(x)))' + \lambda r(x)f(u(x)) = 0 \ \text{ in } (a, b), \tag{4a}$$
$$u(a) = u(b) = 0, \tag{4b}$$

where $\lambda > 0$ is a constant. Under the following assumptions:
$(B_1)$ $f \in C(\mathbb{R})$ is odd and satisfies $tf(t) > 0$ for $t \neq 0$.
$(B_2)$ $r : [a, b] \to (0, +\infty)$ is a continuous function.
$(B_3)$ $\psi$ is odd, increasing, and sub-multiplicative on $[0, +\infty)$, and $\frac{1}{\psi(t)}$ is convex in $t > 0$.
the authors established a Lyapunov-type inequality for (4), having the following result:
**Theorem B** Suppose that conditions $(B_1) - (B_3)$ are satisfied. If $u$ is a nontrivial solution of problem (4), satisfying $u(x) \neq 0$ for $x \in (a, b)$, then the following inequality holds:

$$\frac{2}{\psi\left(\frac{b-a}{2}\right)} \leq \lambda \int_a^b \frac{f(u(x))}{\psi(u(x))}\mathrm{d}x,$$

when the integral exists.

The convexity of $t\psi(t)$ (or $\frac{1}{\psi(t)}$ and sub-multiplicative property of $\psi$) plays an essential role in the establishment of a Lyapunov-type inequality in [12] (or [16]). Our motivation for this paper comes from the papers of [12] and [16]. The main novelty of this paper is to establish Lyapunov-type inequalities for a large class of nonlinear equations governed by (5) (or (4)) without any convexity assumption on $t\psi(t)$ or $\frac{1}{\psi(t)}$, and without any sub-multiplicative assumption on $\psi$. The function $\psi$ in this paper permits much more nonlinearities than that in [12,16] (see e.g., Remark 2 in Section 1 and examples in Section 4). Moreover, under the assumption $(H_4)$, we do not require any odd-even properties of $f$, and require less sign conditions of $f$ than that in [16].

The rest of this paper is organized as follows. Section 2 presents the considered problem and the main results on Lyapunov-type inequalities. Some remarks on the structural conditions are also provided in this section. Detailed proofs of Lyapunov-type inequalities are given in Section 3. Additional examples satisfying our structural conditions are provided in Section 4.

## 2 Problem setting and main results

In this paper, we establish Lyapunov-type inequalities for the following equation

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0 \ \text{ in } (a, b), \tag{5a}$$
$$u(a) = u(b) = 0, \tag{5b}$$

where $\psi$ and $f$ satisfy the following structural conditions having general nonlinearities:

$(H_1)$ $\psi, f \in C((-\infty, \infty)) \cap C^1((0, \infty))$ with $f \not\equiv 0$ on $(-\infty, \infty)$.
$(H_2)$ $\psi$ is odd on $(-\infty, \infty)$.

$(H_3)$ $f(t) \geq 0$ for all $t \in [0, \infty)$.
$(H_4)$ There exists $k_0 > 0$ such that $|f(t)| \leq k_0 \psi(|t|)$ for all $t \in (-\infty, \infty)$.

We make further assumption on $\psi$ or $f$:

$(H_\psi)$ There exists constants $\delta_0, \delta_1 \geq 0$ such that

$$\delta_0 \psi(t) \leq t\psi'(t) \leq \delta_1 \psi(t), \ \ \forall t > 0.$$

$(H_f)$ There exists constants $\theta_0, \theta_1 \geq 0$ such that

$$\theta_0 f(t) \leq t f'(t) \leq \theta_1 f(t), \ \ \forall t > 0.$$

Throughout this paper, we always assume that $r \in L^1(a, b)$ with $r \not\equiv 0$ on $(a, b)$, and conditions $(H_1) - (H_4)$ are satisfied. Moreover, we always assume that (5) has a non-trivial solution $u$ in the sense that $u \in C^1(a, b) \cap C([a, b])$, $\psi(u'(x))$ is absolutely continuous in $x$, and $u$ satisfies the equation in (5) almost everywhere in $(a, b)$.

The first main result is as follows, which can be seen as a complement of the work of [12] and [16] in the setting of functions satisfying $(H_\psi)$ or $(H_f)$.

**Theorem 1** *(i) If $\psi$ satisfies $(H_\psi)$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\delta_0}{1+\delta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$.
*(ii) If $f$ satisfies $(H_f)$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\theta_0}{1+\theta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

We present some corollaries of Theorem 1.

**Corollary 2** *(i) If $\psi(t) = f(t) = |t|^{p-2}t$ $(p > 1)$, then* $\int_a^b |r(x)|dx \geq \frac{2^p}{(b-a)^{p-1}}$, *which is one of the results obtained independently in [9,13].*
*(ii) If $\psi(t) = f(t) = |t|^{a-1}t \log_c(b|t|+d)$, $a, b > 0, c, d > 1$, then* $\int_a^b |r(x)|dx \geq \frac{2(1+a)\ln d}{1+(1+a)\ln d} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a+\frac{1}{\ln d}}\right\}$.
*(iii) If $\psi(t) = f(t) = \frac{|t|^{a-1}t}{\log_c(b|t|+d)}$, $b > 0, c, d > 1, a > \frac{1}{\ln d}$, then* $\int_a^b |r(x)|dx \geq \frac{2(1+a)\ln d}{(1+a)\ln d - 1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a-\frac{1}{\ln d}}\right\}$.

If we make further assumption that $\psi(t)t$ (or $f(t)t$) is convex on $[0 + \infty)$, we get some results stronger than Theorem 1, which can be seen as extensions of [12].

**Theorem 3** *Assume further that $\psi(t)t$ is convex in $t \in [0, +\infty)$.*

*(i) If $\psi$ satisfies $(H_\psi)$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$.
*(ii) If $f$ satisfies $(H_f)$, then* $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

Before the proof of main results, we give some remarks on the structural conditions.

**Remark 1** *(i) We point out $(H_\psi)$ (or $(H_f)$) is a slight version of Lieberman's in [10], where regularity theory was considered for a class of elliptic partial differential equations with a structural condition described by $0 < \delta_0 \leq \frac{t\psi'(t)}{\psi(t)} \leq \delta_1$. It should be noticed that $\frac{t\psi'(t)}{\psi(t)}$ is always required to be positive in [10], while in this paper $\psi'(t)$ (or $f'(t)$) in $(H_\psi)$ (or $(H_f)$) can be zero at some point $t_0 > 0$, i.e., $\delta_0 = 0$ (or $\theta_0 = 0$). Indeed, considering* $\psi(t) = \begin{cases} \frac{7}{8} \cdot (2t)^{\frac{3}{7}}, \ 0 < t < \frac{1}{2}, \\ (t-1)^3 + 1, \ \frac{1}{2} \leq t \leq \frac{3}{2}, \\ \frac{3}{4}t, \ t > \frac{3}{2}, \end{cases}$
*we have $\psi \in C^1((0, +\infty))$ and $0 \leq \frac{t\psi'(t)}{\psi(t)} \leq 3$ for all $t > 0$. The lower boundedness can be achieved when $\psi'(1) = 3(t-1)^2\big|_{t=1} = 0$.*
*(ii) By $(H_1) - (H_4)$, $\psi(0) = f(0) = 0$ and $\psi(t) \geq 0$ for any $t \geq 0$. Furthermore, if $\psi$ (or $f$) satisfies $(H_\psi)$ (or $(H_f)$), then $\psi'(t) \geq 0$ (or $f'(t) \geq 0$), which guarantees the increasing monotonicity of $\psi(t)$ (or $f(t)$) in $t \geq 0$.*

**Remark 2** *In this remark, we give two examples of the function $\psi$ (or $f$) showing that our assumptions on $\psi$ (or $f$) are much weaker than that in [12,16] (see Theorem A and B) in a certain sense. More examples of functions satisfying $(H_\psi)$ or $(H_f)$ are provided in Section 4.*

*(i)* *Let $\psi_0(t)=t+\frac{1}{t}$ with $t\in[\frac{6}{5},\sqrt{3}]$. Due to continuities of $\psi_0$ and $\psi_0'$, there exists $\delta_0',\delta_1'>0$ such that $\delta_0'\le\frac{t\psi_0'(t)}{\psi_0(t)}\le\delta_1'$ for all $t\in[\frac{6}{5},\sqrt{3}]$. Let $p=\frac{11}{61}$, $a=\left(\frac{6}{5}+\frac{5}{6}\right)\left(\frac{6}{5}\right)^{-p}$ and $q=\frac{1}{2}$, $b=\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)(\sqrt{3})^{-q}$. Let $\psi(t)=\begin{cases}at^p, & 0<t<\dfrac{6}{5}\\ \psi_0(t), & \dfrac{6}{5}\le t\le\sqrt{3}\\ bt^q, & t>\sqrt{3}\end{cases}$.*
*By direct computations, one may verify that $\left(\frac{1}{\psi(t)}\right)''=\left(\frac{1}{\psi_0(t)}\right)''=\frac{-2}{(t+1)^3}<0$ for $t\in(\frac{6}{5},\sqrt{3})$. Thus $\frac{1}{\psi}$ is not convex on $[0,+\infty)$, i.e, $\psi$ does not satisfy the condition $(H_3)$ in [16]. However, $\psi$ defined as above satisfies $(H_\psi)$ with $\delta_0=\min\{\delta_0',p,q\}$ and $\delta_1=\max\{\delta_1',p,q\}$ in this paper.*

*(ii)* *Let $\psi_0(t)=\sin t$ with $t\in(0,\frac{\pi}{2})$. Then $(\psi_0(t)t)''=-t\sin t+2\cos t\to-\frac{\pi}{2}<0$ as $t\to\left(\frac{\pi}{2}\right)^-$. Thus there exists $[t_0,t_1]\subset(0,\frac{\pi}{2})$ such that $(\psi_0(t)t)''<0$ for all $t\in[t_0,t_1]$. Let $\psi(t)=\begin{cases}at^p, & 0<t<t_0\\ \psi_0(t), & t_0\le t\le t_1\\ bt^q, & t>t_1\end{cases}$, where $p=t_0\cot t_0>0$, $a=t_0^{-p}\sin t_0>0$ and $q=t_1\cot t_1>0$, $b=t_1^{-q}\sin t_1>0$. Note that there exists $\delta_0',\delta_1'>0$ such that $\delta_0'\le\frac{t\psi_0'(t)}{\psi_0(t)}\le\delta_1'$ for all $t\in[t_0,t_1]$. One may verify that $\psi$ satisfies $(H_\psi)$ with $\delta_0=\min\{\delta_0',p,q\}$ and $\delta_1=\max\{\delta_1',p,q\}$. However, $(\psi(t)t)''<0$ for $t\in[t_0,t_1]$. That is to say such a $\psi$ does not satisfy the convexity condition in [12] while it still satisfies $(H_\psi)$ in this paper.*

## 3 Proof of main results

We present first some auxiliary results needed in the main proof. Let $\Psi(t)=\int_0^t\psi(s)\mathrm{d}s$ for $t\ge0$.

**Lemma 4** *Assume that $\psi$ satisfies $(H_1)$-$(H_4)$ and $(H_\psi)$. The following results hold true.*

*(i)* $\psi(st)\le\max\{s^{\delta_0},s^{\delta_1}\}\psi(t),\ \forall s,t\ge0.$
*(ii)* *$\Psi$ is $C^2-$contiuous on $(0,+\infty)$, and convex on $[0,+\infty)$.*
*(iii)* $\frac{t\psi(t)}{1+\delta_1}\le\Psi(t)\le\frac{t\psi(t)}{1+\delta_0},\ \forall t\ge0.$

**Proof.** Let $h_0(t)=\frac{\psi(t)}{t^{\delta_0}}$, $h_1(t)=\frac{\psi(t)}{t^{\delta_1}}$ for $t>0$. By $(H_\psi)$, it follows

$$h_0'(t)=\frac{\psi'(t)t^{\delta_0}-\psi(t)\delta_0t^{\delta_0-1}}{t^{2\delta_0}}=\frac{t\psi'(t)-\psi(t)\delta_0}{t^{\delta_0+1}}\ge0,$$

which implies that $h_0(t)$ is increasing in $t>0$. Therefore $h_0(st)\le h_0(t)$ for $0\le s\le1$. It follows that

$$\psi(st)\le s^{\delta_0}\psi(t),\ \forall t>0,0\le s\le1. \tag{6}$$

Similarly, one may prove that $h_1(t)$ is decreasing in $t>0$. Then $h_1(st)\le h_1(t)$ for $s\ge1$. It follows that

$$\psi(st)\le s^{\delta_1}\psi(t),\ \forall t>0,s\ge1. \tag{7}$$

By (6) and (7), we have

$$\psi(st)\le\max\{s^{\delta_0},s^{\delta_1}\}\psi(t),\ \forall t>0,s\ge0,$$

which and the continuity of $\psi$ in $t=0$ yields (i).

(ii) is obvious since $\Psi''(t)=\psi'(t)\ge0$ for $t>0$ (see Remark 1) and $\Psi(t)$ is continuous in $t=0$.

To conclude (iii), let $\Psi_0(t) = (1+\delta_0)\Psi(t) - t\psi(t)$ and $\Psi_1(t) = (1+\delta_1)\Psi(t) - t\psi(t)$ for $t \geq 0$. It is easy to see that $\Psi_0'(t) \leq 0$ and $\Psi_1'(t) \geq 0$ for $t > 0$. Then $\Psi_0(t) \leq \Psi_0(0) = 0$ and $\Psi_1(t) \geq \Psi_1(0) = 0$, which and continuities of $\Psi_0, \Psi_1$ yield (iii). ∎

**Remark 3** *Let $F(t) = \int_0^t f(s)ds$ for $t \geq 0$. Then the function $f$ satisfying $(H_1)$-$(H_4)$ and $(H_f)$ and the function $F$ have similar properties as above.*

**Proof of Theorem 1.** Without loss of generality, assume that $|u(c)| = \max\limits_{x\in[a,b]} |u(x)| > 0$ with $c \in (a,b)$. Note that $|u(c)| = \frac{1}{2}\left(\left|\int_a^c u'(x)\mathrm{d}x\right| + \left|\int_c^b u'(x)\mathrm{d}x\right|\right) \leq \frac{1}{2}\int_a^b |u'(x)|\mathrm{d}x$.

Firstly, we prove Theorem 1 (i) under the assumption that $\psi$ satisfies the structural condition $(H_\psi)$. Indeed, by the monotonicity of $\psi$, and Lemma 4 (i) and (iii), we get

$$
\begin{aligned}
\psi(|u(c)|)|u(c)| &\leq \frac{1}{2}\cdot\psi\left(\frac{1}{2}\int_a^b |u'(x)|\mathrm{d}x\right)\cdot\int_a^b |u'(x)|\mathrm{d}x \\
&= \frac{b-a}{2}\cdot\psi\left(\frac{b-a}{2}\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\right)\cdot\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x \\
&\leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\}\cdot\psi\left(\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\right)\cdot\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x \\
&\leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\}\cdot(1+\delta_1)\cdot\Psi\left(\frac{1}{b-a}\int_a^b |u'(x)|\mathrm{d}x\right) \\
&\leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\}\cdot(1+\delta_1)\cdot\frac{1}{b-a}\cdot\int_a^b \Psi(|u'(x)|)\mathrm{d}x \\
&= (1+\delta_1)\cdot\max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}}, \frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\cdot\int_a^b \Psi(|u'(x)|)\mathrm{d}x,
\end{aligned}
\tag{8}
$$

where in the last inequality we used the convexity of $\Psi$ (see Lemma 4 (ii)).
Multiplying (5) by $u$, integrating over $(a,b)$, and using Lemma 4 (iii), $(H_2)$-$(H_4)$, and (8), we get

$$
\begin{aligned}
\int_a^b \Psi(|u'|)\mathrm{d}x &\leq \frac{1}{1+\delta_0}\int_a^b \psi(|u'|)|u'|\mathrm{d}x \\
&= \frac{1}{1+\delta_0}\int_a^b \psi(u')u'\mathrm{d}x \\
&= \frac{1}{1+\delta_0}\int_a^b r(x)f(u)u\mathrm{d}x \\
&\leq \frac{1}{1+\delta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b |r(x)|\mathrm{d}x \\
&\leq \frac{k_0}{1+\delta_0}\max_{x\in[a,b]}(\psi(|u|)|u|)\int_a^b |r(x)|\mathrm{d}x \\
&\leq \frac{k_0}{1+\delta_0}\psi(|u(c)|)|u(c)|\int_a^b |r(x)|\mathrm{d}x \\
&\leq \frac{k_0(1+\delta_1)}{1+\delta_0}\cdot\max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}}, \frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\cdot\int_a^b \Psi(|u'|)\mathrm{d}x\cdot\int_a^b |r(x)|\mathrm{d}x.
\end{aligned}
$$

Note that $\int_a^b \Psi(|u'|)\mathrm{d}x > 0$, otherwise, $\int_a^b \Psi(|u'|)\mathrm{d}x = 0$. By (8) and Lemma 4 (i), we have,

$$
0 \leq \psi(t)|u(c)| = \psi\left(\frac{t}{u(c)}u(c)\right)|u(c)| \leq \max\left\{\left(\frac{t}{u(c)}\right)^{\delta_0}, \left(\frac{t}{u(c)}\right)^{\delta_1}\right\}\psi(u(c))|u(c)| \leq 0, \ \ \forall t \geq 0,
$$

which implies $\psi \equiv 0$ for all $t \in [0,+\infty)$. Then by the odd property of $\psi$, we have $\psi \equiv 0$ for all $t \in (-\infty,+\infty)$. Due to $(H_4)$, $f \equiv 0$ for all $t \in (-\infty,+\infty)$, which is a contradiction with the assumption $(H_1)$.

Then we get

$$\int_a^b |r(x)|\mathrm{d}x \geq \frac{1+\delta_0}{k_0(1+\delta_1)} \cdot \min\left\{\frac{2^{1+\delta_0}}{(b-a)^{\delta_0}}, \frac{2^{1+\delta_1}}{(b-a)^{\delta_1}}\right\}.$$

Theorem 1 (i) has been proven.
Now for Theorem 1 (ii), we proceed in a similar way as above. Indeed, if $f$ satisfies the structural condition $(H_f)$, proceeding as in (8), we get

$$f(|u_c|)|u_c| \leq (1+\theta_1) \cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\} \cdot \int_a^b F(|u'(x)|)\mathrm{d}x. \tag{9}$$

Multiplying (5) by $u$, integrating over $(a,b)$, and using Lemma 4 (iii), $(H_2)$-$(H_4)$, and (9), we get

$$\begin{aligned}
\int_a^b F(|u'|)\mathrm{d}x &\leq \frac{1}{1+\theta_0}\int_a^b f(|u'|)|u'|\mathrm{d}x \\
&\leq \frac{k_0}{1+\theta_0}\int_a^b \psi(|u'|)|u'|\mathrm{d}x \\
&= \frac{k_0}{1+\theta_0}\int_a^b \psi(u')u'\mathrm{d}x \\
&= \frac{k_0}{1+\theta_0}\int_a^b r(x)f(u)u\mathrm{d}x \\
&\leq \frac{k_0}{1+\theta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b |r(x)|\mathrm{d}x \\
&\leq \frac{k_0}{1+\theta_0} f(|u(c)|)|u(c)|\int_a^b |r(x)|\mathrm{d}x \\
&\leq \frac{k_0(1+\theta_1)}{1+\theta_0} \cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\} \cdot \int_a^b F(|u'|)\mathrm{d}x \cdot \int_a^b |r(x)|\mathrm{d}x.
\end{aligned} \tag{10}$$

Note that $\int_a^b F(|u'|)\mathrm{d}x > 0$, otherwise, we can argue as in the proof of (i) to conclude $f(t) \equiv 0$ for any $t \in (-\infty,+\infty)$. Finally, (10) implies the desired result. ■

**Proof of Theorem 3.** The proof of Theorem 3 is a slight modifications of the proof of Theorem 1. Indeed, let $\Phi(t) = \psi(t)t$ for $t \geq 0$, and let $c, u(c)$ be defined as in the proof of Theorem 1. If $\psi$ satisfies the structural condition $(H_\psi)$, arguing as in (8), we get

$$\begin{aligned}
\psi(|u(c)|)|u(c)| &\leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \psi\left(\frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\right) \cdot \frac{1}{b-a}\int_a^b |u'|\mathrm{d}x \\
&= \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \Phi\left(\frac{1}{b-a}\int_a^b |u'|\mathrm{d}x\right) \\
&\leq \max\left\{\left(\frac{b-a}{2}\right)^{1+\delta_0}, \left(\frac{b-a}{2}\right)^{1+\delta_1}\right\} \cdot \frac{1}{b-a}\int_a^b \Phi(|u'|)\mathrm{d}x,
\end{aligned} \tag{11}$$

where in the last inequality we used the convexity of $\Phi$.
Concerning with (5) and (11), we have

$$\int_a^b \Phi(|u'|)\mathrm{d}x = \int_a^b \psi(|u'|)|u'|\mathrm{d}x$$

$$
\begin{aligned}
&= \int_a^b \psi(u')u'\mathrm{d}x \\
&= \int_a^b r(x)f(u)u\mathrm{d}x \\
&\le k_0\psi(|u(c)|)|u(c)| \int_a^b |r(x)|\mathrm{d}x \\
&\le \frac{k_0}{2}\max\left\{\left(\frac{b-a}{2}\right)^{\delta_0}, \left(\frac{b-a}{2}\right)^{\delta_1}\right\} \cdot \int_a^b \Phi(|u'|)\mathrm{d}x \cdot \int_a^b |r(x)|\mathrm{d}x,
\end{aligned}
$$

which yields the desired result in Theorem 3 (i). The desired result in Theorem 3 (ii) can be proven in a similar way. ■

**Proof of Corollary 2.** For (i), it should be noticed that $\delta_0 = \theta_0 = \delta_1 = \theta_1 = p-1$ in $(H_\psi)$ and $(H_f)$).

For (ii), it should be noticed that for $t \ge 0$,

$$\psi(t) = f(t) = t^a \log_c(bt+d),\ a, b > 0, c, d > 1.$$

Then we have

$$a \le \frac{t\psi'(t)}{\psi(t)} = a + \frac{bt}{(bt+d)\ln c \cdot \log_c(bt+d)} \le a + \frac{bt}{(bt+d)\ln c \cdot \log_c d} = a + \frac{1}{\ln d},\quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = a > 0, \delta_1 = \theta_1 = a + \frac{1}{\ln d} > 0$ in $(H_\psi)$ and $(H_f)$.

For (iii), it should be noticed that for $t \ge 0$,

$$\psi(t) = f(t) = \frac{t^a}{\log_c(bt+d)},\ b > 0, c, d > 1, a > \frac{1}{\ln d}.$$

Then we have

$$a - \frac{1}{\ln d} \le \frac{t\psi'(t)}{\psi(t)} = a - \frac{bt}{(bt+d)\ln(bt+d)} \le a,\quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = a - \frac{1}{\ln d} > 0, \delta_1 = \theta_1 = a > 0$ in $(H_\psi)$ and $(H_f)$. ■

## 4 Examples

In this part, we give additional examples of $\psi(t)$ (or $f(t)$) satisfying $(H_\psi)$ (or $(H_f)$), and find $\delta_0, \delta_1$ (or $\theta_0, \theta_1$). For simplicity, we only restrict $\psi(t)$ (or $f(t)$) to the case $t \in [0, +\infty)$, since one may construct functions by odd or even extensions to $t \in (-\infty, +\infty)$.

**Example 1**

$$\psi(t) = f(t) = \ln(1+at) + bt,\quad \forall a > 0, b > 0. \tag{12}$$

*For* (12)*, we have*

$$\frac{t\psi'(t)}{\psi(t)} = \frac{at}{1+at}\frac{1}{\ln(1+at)+bt} + \frac{bt}{\ln(1+at)+bt} \le \frac{a}{b} + 1,\quad \forall t > 0.$$

*Note that* $\ln(1+at) \leq at$ *for all* $t>0$, *it follows*

$$\frac{b}{a+b} \leq \frac{t\psi'(t)}{\psi(t)} \leq \frac{a}{b}+1, \quad \forall t>0.$$

*Thus* $\delta_0 = \theta_0 = \frac{b}{a+b} > 0, \delta_1 = \theta_1 = \frac{a}{b}+1 > 0$ *in* $(H_\psi)$ *and* $(H_f)$. ■

**Example 2**

$$\psi(t) = f(t) = (1+t)\ln(1+t) - t. \tag{13}$$

*For* (13), *firstly, note that* $\psi'(t) = \ln(1+t) \geq 0$ *for any* $t \geq 0$. *Thus* $\psi(t) \geq \psi(0) = 0$. *By direct computations, we have*

$$\frac{t\psi'(t)}{\psi(t)} = \frac{t\ln(1+t)}{(1+t)\ln(1+t)-t} = \frac{t}{(1+t) - \frac{t}{\ln(1+t)}}, \quad \forall t>0.$$

*Since* $\ln(1+t) \leq t$ *for all* $t>0$, *it follows*

$$\frac{t}{(1+t)-\frac{t}{\ln(1+t)}} \geq \frac{t}{(1+t)-\frac{t}{t}} = 1, \quad \forall t>0.$$

*In the following, we prove that for any* $t>0$, *there holds*

$$\frac{t\ln(1+t)}{(1+t)\ln(1+t)-t} \leq 2. \tag{14}$$

*Indeed, let* $h_1(t) = t\ln(1+t) - 2((1+t)\ln(1+t)-t) = 2t - t\ln(1+t) - 2\ln(1+t)$. *Then* $h_1'(t) = 1 - \ln(1+t) - \frac{1}{1+t}$. *Let* $h_2(t) = (1+t) - (1+t)\ln(1+t) - 1 = t - (1+t)\ln(1+t)$. *It is easy to check that* $h_2'(t) = -\ln(1+t) < 0$ *for any* $t>0$. *Thus* $h_2(t) \leq h_2(0) = 0$, *which leads to* $h_1'(t) \leq 0$ *for any* $t>0$. *Therefore* $h_1(t) \leq h_1(0) = 0$. *As a consequence,* (14) *holds true for any* $t>0$. *Finally* $\delta_0 = \theta_0 = 1, \delta_1 = \theta_1 = 2$ *in* $(H_\psi)$ *and* $(H_f)$. ■

**Example 3**

$$\psi(t) = f(t) = \begin{cases} at^p, 0 \leq t < t_0, \\ bt^q + c, t \geq t_0, \end{cases} \tag{15}$$

*where* $a, b, c, p, q, t_0 > 0$ *such that* $at_0^p = bt_0^q + c$, *and* $apt_0^{p-1} = bqt_0^{q-1}$.

*For* (15), *we have* $\psi = f \in C^1((0,+\infty))$ *and*

$$\min\{p,q\} \leq \frac{t\psi'(t)}{\psi(t)} \leq \max\{p,q\}.$$

*Thus* $\delta_0 = \theta_0 = \min\{p,q\} > 0, \delta_1 = \theta_1 = \max\{p,q\} > 0$ *in* $(H_\psi)$ *and* $(H_f)$. ■

**Example 4** *The following example is interesting since* $\psi$ *or* $f$ *is with a variable exponent:*

$$\psi(t) = f(t) = \begin{cases} at^p, 0 \leq t < t_0, \\ bt^{g(t)-1}, t \geq t_0, \end{cases}$$

*where* $t_0 > 1$, $a, b, p > 0$, *and the function* $g \in C^1([t_0,+\infty)$ *satisfy*

$$\begin{cases} c \leq g'(t)t\ln t + g(t) - 1 \leq d, \forall t \geq t_0, \\ p = g'(t_0)t_0\ln t_0 + g(t_0) - 1, \\ a = bt_0^{g(t_0)-1-p}, \end{cases}$$

*with some constants $d \geq c > 0$. Note that $\frac{t(bt^{g(t)-1})'}{bt^{g(t)-1}} = tg'(t)\ln t + g(t) - 1$. By direct computations, one may verify that $\psi = f \in C^1((0,+\infty))$ and satisfy $(H_\psi)$ and $(H_f)$ with $\delta_0 = \theta_0 = \min\{p,c\} = c, \delta_1 = \theta_1 = \max\{p,d\} = d$.* ■

**Example 5** *In [16], the authors provided two examples of $\psi(t)$, i.e,*

*(i)* $\psi(t) = |t|^a \varphi_p(t)$ *with* $a > 1-p$*, and*
*(ii)* $\psi(t) = (\ln(|t|+b))^b \varphi_p(t)$ *with* $a \geq e, b > 0$*,*

*showing that $\frac{1}{\psi(t)}$ is convex in $t > 0$, where $\varphi_p(t) = |t|^{p-2}t$ $(p > 1)$. We point out that $\psi(t)$ given by (i) or (ii) also satisfies the structural condition $(H_\psi)$. Indeed, for (i), it is easy to see that $\frac{t\psi'(t)}{\psi(t)} = a+p-1 > 0$ for $t > 0$. Thus $\delta_0 = \delta_1 = a+p-1$ in $(H_\psi)$. For (ii), by direct computations, we have $\frac{t\psi'(t)}{\psi(t)} = p + \frac{bt}{(t+a)\ln(t+a)}$ for $t > 0$. Note that $0 \leq \frac{bt}{(t+a)\ln(t+a)} \leq \frac{b}{a\ln a}$ for $t > 0$. Then $\delta_0 = p, \delta_1 = p + \frac{b}{a\ln a}$ in $(H_\psi)$.* ■

## References


[1] R. P. Agarwal and A. Özbekler. Lyapunov type inequalities for second order sub and super-half-linear differential equations. *Dynamic Systems & Applications*, 24(1):211–220, 2015.

[2] P. Almenar and L. Jódar. A Lyapunov inequality for a second order nonlinear differential equation. *Appl. Math. Lett.*, 24(4):524–527, 2011.

[3] R. C. Brown and D. B. Hinton. *Lyapunov inequalities and their applications, in: T. M. Rassias (Ed.), Survey on Classical Inequalities*. Kluwer AcademicPublishers, Dordrecht, The Netherlands, 2000.

[4] D. Cakmak. On Lyapunov-type inequality for a class of nonlinear systems. *Electronic Journal of Qualitative Theory of Differential Equations*, 16(1):101–108, 2013.

[5] D. Cakmak. On Lyapunov-type inequality for a class of quasilinear systems. *Electronic Journal of Qualitative Theory of Differential Equations*, 2014(9):1–10, 2014.

[6] S. Cheng. Lyapunov inequalities for differential and difference equations. *Fasc. Math.*, 23(23):25–41, 1991.

[7] O. Dosly and P. Rehak. *Half-Linear Differential Equations, in: Math. Stud.*, volume 202. Amsterdam: Elsevier, North-Holland, 2005.

[8] G. Guseinov and B. Kaymakcalan. Lyapunov inequalities for discrete linear Hamiltonian systems. *Comput. Math. Appl.*, 45(6):1399–1416, 2003.

[9] C. Lee, C. Yeh, C. Hong, and R. P. Agarwal. Lyapunov and Wirtinger inequalities. *Appl. Math. Lett.*, 17(7):847–853, 2004.

[10] G. M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and UralǏtseva for elliptic equations. *Communications in Partial Differential Equations*, 16(2-3):311–361, 1991.

[11] A. Lyapunov. *Probleme General de la Stabilite du Mouvement, in: Ann. Math. Studies*, volume 17. Princeton Univ. Press, 1949. Reprinted from Ann. Fac. Sci. Toulouse, 9 (1907) 203-474, Translation of the original paper published in Comm. Soc. Math. Kharkow, 1892.

[12] P. L. De Napoli and J. P. Pinasco. A Lyapunov inequality for monotone quasilinear operators. *Differential Integral Equations*, 18(10):1193–1200, 2005.

[13] J. P. Pinasco. Lower bounds for eigenvalues of the one-dimensional $p$-laplacian. *Abstr. Appl. Anal.*, 2004(2):147–153, 2004.

[14] J. P. Pinasco. Comparison of eigenvalues for the $p$-laplacian with integral inequalities. *Appl. Math. Comput.*, 182(2):1399–1404, 2006.

[15] J. P. Pinasco. *Lyapunov-type Inequalities With Applications to Eigenvalue Problems, in: Springer Briefs in Mathematics*. Springer, New York-Heidelberg-Dordrecht-London, 2013.

[16] J. Sánchez and V. Vergara. A Lyapunov-type inequality for a $\psi$-laplacian operator. *Nonlinear Analysis*, 74(18):7071–7077, 2011.

[17] X. Tang and M. Zhang. Lyapunov inequalities and stability for linear Hamiltonian systems. *J. Differential Equations*, 252(1):358–381, 2012.

[18] A. Tiryaki. Recent development of Lyapunov-type inequalities. *Adv. Dyn. Syst. Appl.*, 5(2):231–248, 2010.

[19] A. Tiryaki, M. Unal, and D. Cakmak. Lyapunov-type inequalities for nonlinear systems. *J. Math. Anal. Appl*, 332(1):497–511, 2007.